STAT 153 & 248 - Time Series
Lecture Twenty One

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In this lecture, we shall discuss stationarity of AR models. The answer is a bit complicated. Let us start with AR(1) and then consider AR(p) for $p \geq 2$.

1 Stationarity of AR(1)

The AR(1) equation is

$$y_t = \phi_0 + \phi_1 y_{t-1} + \epsilon_t \tag{1}$$

One issue is that this equation does not fully specify y_t and there can be multiple processes $\{y_t\}$ that satisfy (1):

1. Suppose $y_0 = 10$. Define $y_1, y_2, y_3, \dots$ recursively via (1). Also define $y_{-1}, y_{-2}, y_{-3}, \dots$ recursively via the following equation for $t = 0, -1, -2, \dots$:

$$y_{t-1} = -\frac{\phi_0}{\phi_1} + \frac{y_t}{\phi_1} - \frac{\epsilon_t}{\phi_1} \tag{2}$$

   Note that (2) is just a restatement of (1) obtained by rearranging (1) with y_{t-1} on the left hand side. The resulting time series model will then clearly satisfy (1). However it will not be stationary because:

$$\text{var}(y_0) = 0 \quad \text{and} \quad \text{var}(y_1) = \text{var}(\phi_0 + \phi_1 y_0 + \epsilon_1) = \text{var}(\phi_0 + \epsilon_1) = \text{var}(\epsilon_1) = \sigma^2.$$

2. Suppose $|\phi_1| < 1$ and define

$$y_t = \frac{\phi_0}{1 - \phi_1} + \sum_{j=0}^{\infty} \phi_1^j \epsilon_{t-j}. \tag{3}$$

   The summation in the right hand side above is an infinite summation, and hence we need to address convergence issues. Because $|\phi_1| < 1$, the terms ϕ_1^j rapidly decay to 0 as j increases. This ensures that $\sum_{j=0}^{\infty} \phi_1^j \epsilon_{t-j}$ is well-defined.

It is easy to check that (3) satisfies (1) because:

$$\begin{aligned}
y_t &= \frac{\phi_0}{1-\phi_1} + \sum_{j=0}^{\infty} \phi_1^j \epsilon_{t-j} \\
&= \frac{\phi_1}{1-\phi_1} + \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \phi_1^3 \epsilon_{t-3} + \dots \\
&= \frac{\phi_1}{1-\phi_1} + \epsilon_t + \phi_1 \left(\epsilon_{t-1} + \phi_1 \epsilon_{t-2} + \phi_1^2 \epsilon_{t-3} + \dots\right) \\
&= \frac{\phi_1}{1-\phi_1} + \epsilon_t + \phi_1 \left(y_{t-1} - \frac{\phi_0}{1-\phi_1}\right) = \phi_0 + \phi_1 y_{t-1} + \epsilon_t.
\end{aligned}$$

It is also true that (3) is a stationary model. This is because

$$\mathbb{E} y_t = \frac{\phi_0}{1-\phi_1} \qquad \text{for all } t$$

and, for $h \geq 0$,

$$\begin{aligned}
\text{cov}(y_t, y_{t+h}) &= \text{cov}\left(\frac{\phi_0}{1-\phi_1} + \sum_{j=0}^{\infty} \phi_1^j \epsilon_{t-j}, \frac{\phi_0}{1-\phi_1} + \sum_{k=0}^{\infty} \phi_1^k \epsilon_{t+h-k}\right) \\
&= \text{cov}\left(\sum_{j=0}^{\infty} \phi_1^j \epsilon_{t-j}, \sum_{k=0}^{\infty} \phi_1^k \epsilon_{t+h-k}\right) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \phi_1^{j+k} \text{cov}\left(\epsilon_{t-j}, \epsilon_{t+h-k}\right)
\end{aligned}$$

Because $\text{cov}\left(\epsilon_{t-j}, \epsilon_{t+h-k}\right)$ is non-zero (equal to σ^2) only when $t - j = t + h - k$ i.e., $k = j + h$, we get

$$\text{cov}(y_t, y_{t+h}) = \sigma^2 \sum_{j=0}^{\infty} \phi_1^{2j+h} = \sigma^2 \frac{\phi_1^h}{1-\phi_1^2}.$$

This clearly shows that $\{y_t\}$ is stationary with ACVF and ACF given by:

$$\gamma(h) = \sigma^2 \frac{\phi_1^{|h|}}{1-\phi_1^2} \text{ and } \rho(h) = \frac{\gamma(h)}{\gamma(0)} = \phi_1^{|h|}.$$

We have thus proved that (3) is a stationary time series model that satisfies the AR(1) equation (1) when $|\phi_1| < 1$. In fact, it turns out that (3) is the only stationary solution of (1) when $|\phi_1| < 1$ (I am skipping proof of this). Thus (3) is the unique stationary AR(1) model when $|\phi_1| < 1$. The model (3) when $|\phi_1| < 1$ is known as the **causal stationary** AR(1) model. Causal refers to the fact that y_t is fully determined by present and past values of $\{\epsilon_t\}$.

3. Suppose $|\phi_1| > 1$ and define

$$y_t = \frac{\phi_0}{1-\phi_1} - \sum_{j=1}^{\infty} \frac{\epsilon_{t+j}}{\phi_1^j}. \tag{4}$$

Note that y_t is well-defined because the infinite sum above has the coefficients ϕ_1^{-j} which decay rapidly as $|\phi_1| > 1$. It is easy to check that (4) also satisfies the AR(1) equation (1) and is stationary. In fact, it is the unique stationary AR(1) for $|\phi_1| > 1$. The model (4) is called the **non-causal, stationary** AR(1). It is non-causal because y_t depends on the future values of $\epsilon_{t+1}, \epsilon_{t+2}, \dots$.

For the model (4), it is certainly not true that ϵ_t is independent of $y_{t-1}, y_{t-2}, y_{t-3}, \dots$. Recall that we used this estimation while writing the likelihood for AR(1) for parameter estimation. Thus, if we attempt to estimate the parameters ϕ_0, ϕ_1, σ of (4) using our AR-parameter estimation technique, we will get incorrect an estimate of ϕ_1 (for more details, see Example 3.3 and 3.4 in the Shumway-Stoffer book 4th edition).

To summarize the above discussion, there exist many non-stationary AR(1) time series models. When $|\phi_1| < 1$, there exists a unique stationary AR(1) model that is given by the formula (3), this is called the causal, stationary AR(1) model. When $|\phi_1| > 1$, there also exists a unique stationary AR(1) model that is given by the formula (4), this is called the non-causal, stationary AR(1) model.

When $|\phi_1| = 1$ (i.e., $\phi_1 = 1$ or $\phi_1 = -1$), neither of the two formulae (3) and (4) are meaningful (i.e., the infinite series do not converge). Here it turns out that there is no stationary AR(1) model. To see this, consider the case $\phi_1 = 1$ (the case $\phi_1 = -1$ is similar) where

$$y_t = \phi_0 + y_{t-1} + \epsilon_t$$

This implies that for every $t \geq 1$

$$y_t - y_0 = t\phi_0 + \epsilon_1 + \dots + \epsilon_t$$

When $\phi_0 \neq 0$, clearly y_t and y_0 have different means (because $\mathbb{E}y_t = \mathbb{E}y_0 + t\phi_0$) so there is no stationarity. But even if $\phi_0 = 0$, we have

$$\text{var}(y_t - y_0) = \text{var}(\epsilon_1 + \dots + \epsilon_t) = t\sigma^2$$

which approaches ∞ as $t \uparrow \infty$. But if $\{y_t\}$ were stationary, we would have

$$\text{var}(y_t - y_0) \leq 2\text{var}(y_t) + 2\text{var}(y_0) \leq \text{constant}.$$

Thus there are two kinds of AR(1): stationary and non-stationary. Stationarity is only possible when $|\phi_1| \neq 1$. There are also two kinds of stationary AR(1) models. When $|\phi_1| < 1$, the stationary AR(1) model has the formula (3); this is the causal kind of stationarity. When $|\phi_1| > 1$, the stationary AR(1) model has the formula (4); this is the non-causal kind of stationarity.

2 On the formulae for stationary AR(1)

Above, we first wrote down the formulae (3) and (4) for stationary AR(1) and then verified that they indeed satisfy the AR(1) equation. It turns that these formulae can be derived by "solving" the AR(1) equation (1) for y_t in terms of $\{\epsilon_t\}$. We shall present this solution method here. We will not present a rigorous justification for this method (which can be found, for example, in the book "Time Series: theory and methods" by Brockwell and Davis).

Before describing this solution method, we need to introduce the backshift notation.

2.1 Backshift Notation

A convenient piece of notation used while working with AR and MA models is the Backshift notation. Let B denote the *backshift operator* defined by

$$By_t = y_{t-1}, B^2 y_t = y_{t-2}, B^3 y_t = y_{t-3}, \dots$$

and similarly

$$B\epsilon_t = \epsilon_{t-1}, B^2\epsilon_t = \epsilon_{t-2}, B^3\epsilon_t = \epsilon_{t-3}, \dots.$$

Also let I denote the identity operator: $Iy_t = y_t$. More generally, we can define polynomial functions of the Backshift operator by, for example,

$$(I + B + 3B^2)y_t = Iy_t + By_t + 3B^2y_t = y_t + y_{t-1} + 3y_{t-2}.$$

In general, for every polynomial $f(z)$, we can define $f(B)$. One can even extend this notation to negative powers of B which correspond to forward shifts. For example, $B^{-1}y_t = y_{t+1}, B^{-5}y_t = y_{t+5}$ and $(B^3 + 9B^{-2})y_t = y_{t-3} + 9y_{t+2}$ etc.

In this notation, the defining equation $y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \epsilon_t$ for the $AR(p)$ model can be written as $\phi(B)y_t = \phi_0 + \epsilon_t$ for the polynomial $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p$.

The defining equation $y_t = \epsilon_t + \theta\epsilon_{t-1}$ for the MA(1) model can be written as $y_t = \theta(B)\epsilon_t$ for the polynomial $\theta(z) = 1 + \theta_1 z$.

The defining equation $y_t = \epsilon_t + \theta_1\epsilon_{t-1} + \dots + \theta_q\epsilon_{t-q}$ for the MA(q) model becomes $y_t = \theta(B)\epsilon_t$ for the polynomial $\theta(z) = 1 + \theta_1 z + \dots \theta_q z^q$.

2.2 $AR(1)$ solutions using Backshift Calculus

The two stationary solutions (3) and (4) to the AR(1) difference equation (1) for the two cases $|\phi_1| < 1$ and $|\phi_1| > 1$ can also be derived using formal operations that are sometimes known as Backshift Calculus. This is described in this section. First note that (1) can be written as

$$\phi(B)y_t = \phi_0 + \epsilon_t \qquad \text{where } \phi(z) = 1 - \phi_1 z.$$

Thus we can formally write

$$y_t = \frac{1}{\phi(B)}(\phi_0 + \epsilon_t).$$

Using

$$\frac{1}{\phi(z)} = \frac{1}{1 - \phi_1 z} = 1 + \phi_1 z + \phi_1^2 z^2 + \phi_1^3 z^3 + \dots,$$

we obtain

$$\begin{aligned} y_t &= \left(I + \phi_1 B + \phi_1^2 B^2 + \dots\right)(\phi_0 + \epsilon_t) \\ &= \left(I + \phi_1 B + \phi_1^2 B^2 + \dots\right)\phi_0 + \left(I + \phi_1 B + \phi_1^2 B^2 + \dots\right)\epsilon_t \\ &= \left(1 + \phi_1 + \phi_1^2 + \dots\right)\phi_0 + \sum_{j=0}^{\infty}\phi_1^j\epsilon_{t-j} = \frac{\phi_0}{1 - \phi_1} + \sum_{j=0}^{\infty}\phi_1^j\epsilon_{t-j} \end{aligned}$$

which gives (3).

When $|\phi_1| > 1$, the process (3) does not make sense. So we expand $1/\phi(z)$ in the following alternative way:

$$\begin{aligned} \frac{1}{\phi(z)} &= \frac{1}{1 - \phi_1 z} \\ &= \frac{-1}{\phi_1 z}\left(1 - \frac{1}{\phi_1 z}\right)^{-1} \\ &= \frac{-1}{\phi_1 z}\left(1 + \frac{1}{\phi_1 z} + \frac{1}{\phi_1^2 z^2} + \dots\right) = -\frac{z^{-1}}{\phi_1} - \frac{z^{-2}}{\phi_1^2} - \frac{z^{-3}}{\phi_1^3} - \dots. \end{aligned}$$

We thus get

$$
\begin{aligned}
y_t &= \frac{1}{\phi(B)}\left(\phi_0 + \epsilon_t\right) \\
&= \left(-\frac{B^{-1}}{\phi_1} - \frac{B^{-2}}{\phi_1^2} - \frac{B^{-3}}{\phi_1^3} - \dots\right)(\phi_0 + \epsilon_t) = \frac{\phi_0}{1-\phi_1} - \sum_{j=1}^{\infty} \frac{\epsilon_{t+j}}{\phi_1^j}
\end{aligned}
$$

which gives (4). This formal method is called Backshift Calculus and it works for higher order AR models as well.

3 Stationary and Causality for AR(p), $p \geq 2$

Similar to AR(1), it is possible to characterize parameter regimes which ensure existence of stationary (and also causal/non-causal) solutions of AR(p) for $p \geq 1$. Recall that the AR(p) model is given by the equation:

$$y_t = \phi_0 + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \epsilon_t. \tag{5}$$

In terms of the Backshift Notation, we can write the model as:

$$\phi(B) y_t = \phi_0 + \epsilon_t$$

where $\phi(B)$ is the result of the following polynomial applied to the Backshift operator:

$$\phi(z) := 1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p. \tag{6}$$

This polynomial is called the AR(p) polynomial or the AR(p) characteristic polynomial. This polynomial will have p roots $z_1, \dots, z_p$. Some of these roots may be complex.

1. Suppose all the roots z_i have modulus distinct from one: $|z_i| \neq 1$ for every i. Then there exists a unique stationary solution to (5). Backshift calculus can be used to write the stationary solution y_t explicitly in terms of $\{\epsilon_t\}$. We shall see how to do this in the next lecture.
2. Suppose all the roots z_i have modulus strictly larger than one: $|z_i| > 1$ for every i. Then the unique stationary solution is of the form $y_t = \mu + \psi_0 \epsilon_t + \psi_1 \epsilon_{t-1} + \psi_2 \epsilon_{t-j} + \dots = \mu + \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}$, for some μ and $\{\psi_j, j \geq 0\}$. In other words, only the current and past ϵ_t values $(\epsilon_t, \epsilon_{t-1}, \dots)$ determine y_t. Therefore, this stationary solution is causal.
3. Suppose at least one of the roots z_i has modulus strictly smaller than 1 while all other roots have moduli strictly larger than 1. In this case, the unique stationary solution will involve ϵ_t-terms from both in the past and future: $y_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j \epsilon_{t-j}$ (note that the sum is now going from $-\infty$ to ∞). This stationary solution is non-causal.
4. Suppose at least one root z_i has modulus exactly equal to 1. Then there is no stationary solution to (5).

When $p = 1$, the AR(1) polynomial is $\phi(z) = 1 - \phi_1 z$ with root $1/\phi_1$. So the root having magnitude more than 1 is equivalent to $|\phi_1| < 1$. Then the above assertions are equivalent to the assertions made in the previous two sections for AR(1).

We will see more details and examples in the next lecture.

4 Additional Optional Reading

1. Sections 3.1, 3.2, 3.3 of Shumway-Stoffer 4th edition.