STAT 153 & 248 - Time Series
Lecture Seven

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1 Recap from last lecture

In the last couple of lectures, we considered fitting the sinusoidal model

$$y_t = \beta_0 + \beta_1 \cos(2\pi f t) + \beta_2 \sin(2\pi f t) + \epsilon_t \qquad \text{with } \epsilon_t \overset{\text{i.i.d}}{\sim} N(0, \sigma^2) \tag{1}$$

to observed time series $y_1, \ldots, y_n$. A key role for inferring the frequency parameter f in this model is played by:

$$RSS(f) := \min_{\beta_0, \beta_1, \beta_2} \sum_{t=1}^{n} (y_t - \beta_0 - \beta_1 \cos(2\pi f t) - \beta_2 \sin(2\pi f t))^2 = \|y - X_f \hat{\beta}_f\|^2$$

where

$$y = \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{pmatrix} \quad \text{and} \quad X_f = \begin{pmatrix} 1 & \cos(2\pi f(1)) & \sin(2\pi f(1)) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & \cos(2\pi f(n)) & \sin(2\pi f(n)) \end{pmatrix} \quad \text{and} \quad \hat{\beta}_f = (X_f^T X_f)^{-1} X_f^T y.$$

$RSS(f)$ is simply the residual sum of squares in the linear regression model obtained by fixing the frequency parameter f. It describes how well the sinusoid with frequency f fits the observed data $y_1, \ldots, y_n$.

In the last lecture, we derived the following alternative formula for $RSS(f)$ that holds when $f \in (0, 0.5)$ is a Fourier frequency i.e., nf is an integer:

$$RSS(f) = \sum_t (y_t - \bar{y})^2 - 2I(f) \qquad \text{when } f \in (0, 0.5) \text{ is a Fourier Frequency} \tag{2}$$

where $I(f)$ is defined by

$$I(f) := \frac{1}{n} \left(\sum_{t=1}^{n} y_t \cos(2\pi f t) \right)^2 + \frac{1}{n} \left(\sum_{t=1}^{n} y_t \sin(2\pi f t) \right)^2 \qquad \text{for } f \in (0, 0.5)$$

$I(f)$ is known as the *Periodogram* of the data $y_1, \ldots, y_n$.

In this lecture, we study the Discrete Fourier Transform (DFT) of the observed time series data, and see how the DFT is related to the periodogram.

2 Orthogonality Properties of discretely-sampled Sinusoids at Fourier Frequencies

The key fact underlying the Discrete Fourier Transform (DFT) is orthogonality of (discretely sampled) sinusoids at Fourier frequencies. Last lecture, we proved the following formulae. If f is a Fourier frequency with $0 < f < 0.5$, then

$$\sum_{t=1}^{n} \cos(2\pi f t) = 0 \qquad \sum_{t=1}^{n} \sin(2\pi f t) = 0$$
$$\sum_{t=1}^{n} \cos^2(2\pi f t) = \frac{n}{2} \qquad \sum_{t=1}^{n} \sin^2(2\pi f t) = \frac{n}{2}$$
$$\sum_{t=1}^{n} \cos(2\pi f t)\sin(2\pi f t) = 0$$

The sums above are over $t = 1, \ldots, n$. It should be clear that the same formulae are true if we sum over $t = 0, 1, \ldots, n-1$ (because the value of $\cos(2\pi f t)$ and $\sin(2\pi f t)$ at $t = 0$ and $t = n$ coincide when f is a Fourier frequency). When discussing the DFT, it is a standard convention to take $t = 0, 1, \ldots, n-1$, and this is what we shall be doing in the this lecture. Rewriting the above formulae with $t = 0, 1, \ldots, n-1$, we get

$$\sum_{t=0}^{n-1} \cos(2\pi f t) = 0 \qquad \sum_{t=0}^{n-1} \sin(2\pi f t) = 0 \tag{3}$$
$$\sum_{t=0}^{n-1} \cos^2(2\pi f t) = \frac{n}{2} \qquad \sum_{t=0}^{n-1} \sin^2(2\pi f t) = \frac{n}{2} \tag{4}$$
$$\sum_{t=0}^{n-1} \cos(2\pi f t)\sin(2\pi f t) = 0 \tag{5}$$

when $f \in (0, 0.5)$ is a Fourier frequency. The conditions (3) say that discretely sampled sinusoids at Fourier frequencies $f \in (0, 0.5)$ have mean zero. Conditions (4) say that they have energy equal to $n/2$. Condition (7) says that the discretely sampled cosine and sine are orthogonal, or equivalently, that their correlation is zero.

There is another very important property. If f_1 and f_2 are two distinct Fourier frequencies in $(0, 0.5)$, then

$$\begin{aligned} &\sum_{t=0}^{n-1} \cos(2\pi f_1 t)\cos(2\pi f_2 t) = 0 \qquad \sum_{t=0}^{n-1} \cos(2\pi f_1 t)\sin(2\pi f_2 t) = 0 \\ &\sum_{t=0}^{n-1} \cos(2\pi f_1 t)\sin(2\pi f_2 t) = 0 \qquad \sum_{t=0}^{n-1} \cos(2\pi f_1 t)\sin(2\pi f_2 t) = 0 \end{aligned} \tag{6}$$

This means that the sampled sinusoids at distinct Fourier frequencies are orthogonal (or equivalently uncorrelated).

These properties allow us to construct an orthogonal basis for $\mathbb{R}^n$ consisting of sinusoids. Define

$$\mathbf{c}^j := (1, \cos(2\pi j/n), \cos(2\pi 2j/n), \ldots, \cos(2\pi(n-1)j/n))^T$$

and

$$\mathbf{s}^j := (0, \sin(2\pi j/n), \sin(2\pi 2j/n), \ldots, \sin(2\pi(n-1)j/n))^T.$$

These are the vectors obtained by evaulating $\cos(2\pi ft)$ and $\sin(2\pi ft)$ with Fourier Frequency $f = j/n$ at time points $t = 0, 1, \ldots, (n-1)$. So far we have assumed that $0 < f = j/n < 1/2$ i.e., $0 < j < n/2$. When $j = 0$, the vector $\mathbf{c}^0$ is the vector of all ones, while $\mathbf{s}^0$ equals the zero vector. When $f = 1/2$ (this is only when n is even other $1/2$ will not be a Fourier frequency) or $j = n/2$, we have

$$\mathbf{c}^{n/2} = (1, -1, 1, -1, \ldots, (-1)^{n-1}) \quad \text{and} \quad \mathbf{s}^{n/2} = (0, \ldots, 0).$$

Thus when n is even, we are looking at the vectors:

$$\mathbf{c}^0, \mathbf{c}^1, \mathbf{s}^1, \ldots, \mathbf{c}^{\frac{n}{2}-1}, \mathbf{s}^{\frac{n}{2}-1}, \mathbf{c}^{\frac{n}{2}}.$$

When n is odd, we are looking at

$$\mathbf{c}^0, \mathbf{c}^1, \mathbf{s}^1, \ldots, \mathbf{c}^{\frac{n-1}{2}}, \mathbf{s}^{\frac{n-1}{2}}.$$

In either case, the total number of these vectors equals n. By the properties stated above, these vectors are orthogonal, and hence form a basis for the n-dimensional vector space of real-valued vectors $\mathbb{R}^n$. This means that every vector $y \in \mathbb{R}^n$ can be written as a linear combination of these basis vectors. The coefficients in this linear combination are closely related to the DFT. This is the idea behind the DFT. For the formal definition, we use complex exponentials as opposed to sines and cosines.

3 Complex Sinusoidal Vectors at Fourier Frequencies

3.1 $f \in [0, 1)$ as opposed to $f \in [0, 0.5]$

$\cos(2\pi ft)$ and $\sin(2\pi ft)$ can be represented in terms of complex exponentials as

$$\cos(2\pi ft) = \frac{1}{2}\exp(2\pi ift) + \frac{1}{2}\exp(-2\pi ift) \quad \text{and} \quad \sin(2\pi ft) = \frac{1}{2i}\exp(2\pi ift) - \frac{1}{2i}\exp(-2\pi ift)$$

Note here that we have to deal with $-f$ as well (because of the second term $e^{-2\pi ift} = e^{2\pi i(-f)t}$) and $-f$ lies between $-1/2$ and 0. Thus when discussing sinusoids in terms of complex exponentials $e^{2\pi ift}, t = 0, 1, \ldots, n-1$, we take $f \in (-0.5, 0.5]$ (note that $f = -0.5$ leads to the same $e^{2\pi ift}$ as $f = 0.5$ so we drop $f = -0.5$ from consideration). If one does not want to deal with negative frequencies, then we can use

$$e^{-2\pi ift} = \cos(2\pi ft) - i\sin(2\pi ft) = \cos(2\pi(1-f)t) + i\sin(2\pi(1-f)t) = e^{2\pi i(1-f)t}$$

because $\cos(2\pi(1-f)t) = \cos(2\pi t - 2\pi ft) = \cos(2\pi ft)$ (note t is an integer) and $\sin(2\pi(1-f)t) = \sin(2\pi t - 2\pi ft) = -\sin(2\pi ft)$.

Therefore, if we want to use complex exponentials $e^{2\pi ift}$ but we do not want to deal with negative frequencies, then we can restrict f to $[0, 1)$. From here on, whenever we consider the complex sinusoid $x_t = e^{2\pi ift}$ for $t = 0, 1, \ldots, n-1$, we restrict $f \in [0, 1)$.

3.2 Complex Sinusoids

For every $0 \leq j \leq (n-1)$, let us define the $n \times 1$ vector

$$u^j = (1, \exp(2\pi ij/n), \exp(2\pi i2j/n), \ldots, \exp(2\pi i(n-1)j/n))^T.$$

This vector can be interpreted as the complex sinusoid $e^{2\pi ift}$ with Fourier frequency $f = j/n$ evaluated at the time points $t = 0, 1, \ldots, (n-1)$. It is easy to see that

1. When $j = 0$, we have $u^0 = (1, 1, \ldots, 1)$.
2. When $1 \le j \le n-1$, we have $u^j = \overline{u^{n-j}}$. Here $\bar{u}$ denotes complex conjugate of u (the complex conjugate $\bar{u}$ of a vector u is defined as the vector obtained by taking the complex conjugates of each entry of u).

The most important property of these complex valued vectors $u^0, u^1, \ldots, u^{n-1}$ is **orthogonality**. Specifically, for $0 \le j \ne k \le n-1$, we have

$$\left\langle u^j, u^k \right\rangle = 0. \tag{7}$$

Recall that the inner product between two complex valued vectors $a = (a_1, \ldots, a_n)^T$ and $b = (b_1, \ldots, b_n)^T$ is given by

$$\langle a, b \rangle = \sum_{j=1}^{n} a_j \bar{b}_j.$$

Note specially the complex conjugate of b_j above.

Here is the proof of (7). Fix $0 \le j \ne k \le n-1$ and write

$$\begin{aligned}
\left\langle u^j, u^k \right\rangle &= \sum_{t=0}^{n-1} \exp\left(2\pi i \frac{j}{n} t\right) \overline{\exp\left(2\pi i \frac{k}{n} t\right)} \\
&= \sum_{t=0}^{n-1} \exp\left(2\pi i \frac{j}{n} t\right) \exp\left(-2\pi i \frac{k}{n} t\right) \\
&= \sum_{t=0}^{n-1} \exp\left(2\pi i \frac{j-k}{n} t\right) \\
&= \sum_{t=0}^{n-1} \left[\exp\left(2\pi i \frac{j-k}{n}\right)\right]^t \\
&= \frac{1 - \left(\exp\left(2\pi i \frac{j-k}{n}\right)\right)^n}{1 - \exp\left(2\pi i \frac{j-k}{n}\right)} \\
&= \frac{1 - \exp\left(2\pi i (j-k)\right)}{1 - \exp\left(2\pi i \frac{j-k}{n}\right)} \\
&= \frac{1 - \cos\left(2\pi (j-k)\right) - i \sin\left(2\pi (j-k)\right)}{1 - \exp\left(2\pi i \frac{j-k}{n}\right)} = \frac{1 - 1 - 0}{1 - \exp\left(2\pi i \frac{j-k}{n}\right)} = 0
\end{aligned}$$

This proves (7). It is also easy to see that (just take $j = k$ in the above calculation and the answer can be found in the third line)

$$\left\langle u^j, u^j \right\rangle = \|u^j\|^2 = n.$$

Therefore the n complex-valued vectors $u^0, u^1, \ldots, u^{n-1}$ are orthogonal and they all have the same squared length equal to n. This immediately implies that they form a **basis** for the space $\mathbb{C}^n$ consisting of all complex-valued vectors of length n. In other words, every complex-valued vector of length n can be written as a linear combination of $u^0, u^1, \ldots, u^n$. This observation is the foundation for the definition of the Discrete Fourier Transform.

3.3 The Discrete Fourier Transform (DFT)

Because $u^0, \dots, u^{n-1}$ form a basis, we can write any $n \times 1$ vector of complex entries:

$$y = (y_0, \dots, y_{n-1})^T$$

as a linear combination of $u^0, \dots, u^{n-1}$. More specifically, we can write

$$y = a_0 u^0 + a_1 u^1 + \dots + a_{n-1} u^{n-1} \tag{8}$$

Take the inner product of both sides of the above equation with u^j for a fixed j and use orthogonality so that $\langle u^j . u^k \rangle = 0$ for $k \neq j$ and the fact that $\langle u^j, u^j \rangle = n$ to obtain

$$a_j = \frac{1}{n} \langle y, u^j \rangle = \frac{1}{n} \sum_{t=0}^{n-1} y_t \exp\left(-\frac{2\pi i j t}{n}\right). \tag{9}$$

We are now ready to define the Discrete Fourier Transform (DFT). The DFT of $y_0, y_1, \dots, y_{n-1}$ is defined by

$$b_j := \langle y, u^j \rangle = \sum_{t=0}^{n-1} y_t \exp\left(-\frac{2\pi i j t}{n}\right). \tag{10}$$

More specifically, the n (possibly) complex numbers $b_0, b_1, \dots, b_{n-1}$ are collectively called the DFT of $y_0, \dots, y_{n-1}$. Typically, $y_0, \dots, y_{n-1}$ will represent observed time series data. It is important to note that even though $y_0, \dots, y_{n-1}$ are real-valued, their DFT $b_0, \dots, b_{n-1}$ can be complex-valued.

Here are some basic things to note about the DFT:

1. b_0 is always equal to $y_0 + \dots + y_{n-1}$. To see this, just plug in $j = 0$ in (10).
2. In general b_j is a complex number with real and imaginary parts given by:

$$\text{real part of } b_j = \sum_{t=0}^{n-1} y_t \cos\left(\frac{2\pi j t}{n}\right) \quad \text{and} \quad \text{imaginary part of } b_j = -\sum_{t=0}^{n-1} y_t \sin\left(\frac{2\pi j t}{n}\right)$$

3. The DFT of $y = (y_0, \dots, y_{n-1})^T$ can be obtained in `numpy` using the command `np.fft.fft(x)`. Here `fft` stands for Fast Fourier Transform which is a special efficient algorithm for computing the DFT (we will not be going over the details of the FFT algorithm).

3.4 The Periodogram

The Periodogram is a way of visualizing the DFT. The DFT consists of complex numbers so it is difficult to visualize it directly. The common visualization consists of looking at the squared absolute values of the DFT. More precisely, the periodogram is defined by

$$I\left(\frac{j}{n}\right) := \frac{|b_j|^2}{n} \qquad \text{for } 0 < \frac{j}{n} \leq \frac{1}{2}.$$

One visualizes the size of the DFT terms by plotting the periodogram. Note that $j = 0$ is not plotted as b_0 is simply the sum of the data values and does not provide any information on the sinusoidal components present in the data.

Because

$$b_j = \sum_{t=0}^{n-1} y_t \exp\left(-\frac{2\pi ijt}{n}\right) = \sum_{t=0}^{n-1} y_t \cos\frac{2\pi jt}{n} - i\sum_{t=0}^{n-1} y_t \sin\frac{2\pi jt}{n},$$

we can write the periodogram as:

$$I\left(\frac{j}{n}\right) = \frac{1}{n}\left[\left(\sum_{t=0}^{n-1} y_t \cos\frac{2\pi jt}{n}\right)^2 + \left(\sum_{t=0}^{n-1} y_t \sin\frac{2\pi jt}{n}\right)^2\right] \qquad \text{for } 0 < \frac{j}{n} \leq \frac{1}{2}. \tag{11}$$